On the Duality of Quasi-Exactly Solvable Problems

Agnieszka Krajewska[1], **Alexander Ushveridze**[2]
and
Zbigniew Walczak[3]

Department of Theoretical Physics, University of Lodz,
Pomorska 149/153, 90-236 Lodz, Poland

Abstract

It is demonstrated that quasi-exactly solvable models of quantum mechanics admit an interesting duality transformation which changes the form of their potentials and inverts the sign of all the exactly calculable energy levels. This transformation helps one to reveal some new features of quasi-exactly solvable models and associated orthogonal polynomials.

[1]akraj@mvii.uni.lodz.pl
[2]alexush@mvii.uni.lodz.pl
[3]walczak@mvii.uni.lodz.pl

1 Introduction

In this paper we demonstrate that quasi-exactly solvable models of quantum mechanics (i.e. models admitting exact solutions only for certain limited parts of the spectrum)[4] normally appear in the form of dublets. The models forming a given dublet look differently but their exactly calculable energy levels exactly coincide up to a sign transformation. We call such models *dual* to each other. Note that the duality property does not hold for other (exactly noncalculable) levels.

In this paper, we consider simplest quasi-exactly solvable models introduced in ref. [2] and represented by even polynomials of degree six (section 2). We construct the dual pairs of these models (section 3) and show that the duality property enables one to estimate the intervals of the spectra occupied by exactly calculable energy levels (section 4). We also demonstrate that the duality property can be used for finding classical counterparts of the phenomenon of quasi-exact solvability (section 5) and for establishing new remarkable properties of non-standard orthogonal polynomials associated with quasi-exactly solvable models (section 6). A brief discussion of the general case will be given in the concluding section 7.

2 The model

The Schrödinger equation for the simplest sequence of quasi-exactly solvable models reads

$$\left\{-\hbar^2\frac{\partial^2}{\partial x^2} + x^2(ax^2+b)^2 - \hbar a(2M+3)x^2\right\}\Psi(x) = E\Psi(x), \tag{2.1}$$

where b is real, $a > 0$ is a positive real number and $M = 0, 1, \ldots$ is an arbitrary non-negative integer.

The explicit solutions of this equation can be represented in the form

$$E = \hbar\left[(2M+1)b + 2a\sum_{i=1}^{M}\xi_i^2\right] \tag{2.2}$$

and

$$\Psi(x) = \prod_{i=1}^{M}(x-\xi_i)\exp\left[-\frac{1}{\hbar}\left(\frac{bx^2}{2}+\frac{ax^4}{4}\right)\right], \tag{2.3}$$

where the numbers $\xi_i,\ i = 1, \ldots, M$ (playing the role of wavefunction zeros) satisfy the system of numerical equations

$$\sum_{k=1}^{M}{}' \frac{\hbar}{\xi_i - \xi_k} = b\xi_i + a\xi_i^3, \quad i = 1, \ldots, M \tag{2.4}$$

and the following additional condition

$$\sum_{i=1}^{M}\xi_i = 0. \tag{2.5}$$

[4] For details see original papers [1, 2, 3, 4] and reviews [5, 6].

The proof of formulas (2.2) – (2.5) is trivial. For details see the proof of analogous formulas in ref. [6]. The number of exactly calculable energy levels and wavefunctions is, obviously, given by the number of solutions of the system (2.4) – (2.5). At first sight, this system is overdetermined, because it contains $M+1$ equations for only M unknowns ξ_i, $i=1,\ldots,M$. However, it can be shown that, due to the symmetry of this system under reflections $\xi_i \to -\xi_i$, it still has non-trivial solutions for any M, and their number is equal to $[M/2]+1$, where $[\ ]$ is a standard notation for an integer part. Note that the parity of the obtained energy levels coincides with the parity of M. It can also be shown that the numbers ξ_i satisfying the system (2.4) – (2.5) are either real or purely imaginary.

3 Duality and self-duality

It is easily seen that formulas (2.1) – (2.5) for the "plus" model (with $b=|b|$) can be reduced to the same formulas for "minus" model (with $b=-|b|$) after the changes

$$x \to ix, \qquad \xi_i \to i\xi_i, \qquad E \to -E. \tag{3.1}$$

Hereafter we shall call the models (2.1), differing from each other by the sign of the parameter b, *dual* to each other and the transformation (3.1) — the *duality transformation.* We also shall use the notations

$$V(x) = x^2(ax^2+b)^2 - \hbar a(2M+3)x^2, \tag{3.2}$$

for the potential of the original model (2.1), and

$$\bar{V}(x) = x^2(ax^2-b)^2 - \hbar a(2M+3)x^2, \tag{3.3}$$

for its dual. If the parameter b is zero then the potentials $V(x)$ and $\bar{V}(x)$ exactly coincide and read

$$V(x) = \bar{V}(x) = a^2x^6 - \hbar a(2M+3)x^2. \tag{3.4}$$

It is natural to call the model (3.4) *self-dual.*

4 Spectral properties

From formula (3.1) it follows that, if E is an exactly calculable energy level in model (3.2), then $\bar{E}=-E$ is also a certain exactly calculable energy level in the dual model (3.3). It is interesting to establish a relationship between the quantum numbers (ordinal numbers) of these levels. Assume that the energy level E describes the Kth excitation, $E=E_K$. Then, according to the oscillation theorem, the wavefunction (2.3), $\Psi(x)=\Psi_K(x)$, should have K real zeros (nodes) and, therefore, K numbers ξ_i should lie on the real x-axis. In this case, the $M-K$ remaining numbers ξ_i should lie on the imaginary x-axis. But from formula (3.1) it follows that, after the duality transformation, all the imaginary numbers ξ_i describing the imaginary wavefunction zeros in the model (3.2) become real, $\bar{\xi}_i = i\xi_i$, and take the meaning of real wavefunction zeros (nodes) in the dual model (3.3). But this means that $\bar{\Psi}(x)=\bar{\Psi}_{M-K}(x)$ the corresponding energy level describes the $(M-K)$th excitation, $\bar{E}=\bar{E}_{M-K}$. This leads us to the simple relations

$$E_K = -\bar{E}_{M-K}, \qquad \Psi_K(x) \sim \bar{\Psi}_{M-K}(ix). \tag{4.1}$$

In particular, the minimal exactly calculable energy level in a given quasi-exactly solvable model (3.2) coincides with the maximal exactly calculable energy level in the corresponding dual model (3.3) taken with sign minus! For the self-dual model (3.4) the situation becomes even more transparent. In this case, because of the coincidence of the sets $\{E\}$ and $\{\bar{E}\}$, we obtain

$$E_K = -E_{M-K}, \qquad \Psi_K(x) \sim \Psi_{M-K}(ix). \tag{4.2}$$

The abovementioned facts enable one to detemine the intervals of the spectra of models (3.2), (3.3) and (3.4) occupied by the exactly calculable levels. We denote these intervals by $[E_{min}, E_{max}]$ and $[\bar{E}_{min}, \bar{E}_{max}]$, respectively. The duality property means that

$$E_{max} = -\bar{E}_{min}, \qquad \bar{E}_{max} = -E_{min}. \tag{4.3}$$

On the other hand, we have the obvious inequalities

$$E_{min} > \min\, V(x), \qquad \bar{E}_{min} > \min\, \bar{V}(x). \tag{4.4}$$

Comparing (4.3) with (4.4), we obtain

$$E_{max} < -\min\, \bar{V}(x), \qquad \bar{E}_{max} < -\min\, V(x). \tag{4.5}$$

This means that

$$[E_{min}, E_{max}] \subset [\min\, V(x), -\min\, \bar{V}(x)], \quad [\bar{E}_{min}, \bar{E}_{max}] \subset [\min\, \bar{V}(x), -\min\, V(x)], \tag{4.6}$$

where

$$\min\, V(x) = \left\{ \begin{array}{l} 0, \quad b > \sqrt{ac}, \\ 2(27a)^{-1}\left(-2b + \sqrt{b^2 + 3ca}\right)\left(-3ca + b^2 + b\sqrt{b^2 + 3ca}\right), \quad b < \sqrt{ac} \end{array} \right. \tag{4.7}$$

and

$$\min\, \bar{V}(x) = \left\{ \begin{array}{l} 0, \quad -b > \sqrt{ac}, \\ 2(27a)^{-1}\left(+2b + \sqrt{b^2 + 3ca}\right)\left(-3ca + b^2 - b\sqrt{b^2 + 3ca}\right), \quad -b < \sqrt{ac} \end{array} \right. \tag{4.8}$$

Here we introduced a new quantity

$$c = (2M + 3)\hbar. \tag{4.9}$$

For the self-dual model (3.4), formulas (4.6) – (4.8) become simpler. Taking in them $b = 0$, we obtain

$$[E_{min}, E_{max}] = [\bar{E}_{min}, \bar{E}_{max}] \subset \left[-\frac{\sqrt{12a}c^{3/2}}{9}, \frac{\sqrt{12a}c^{3/2}}{9}\right]. \tag{4.10}$$

The examples of the dual and self-dual potentials with the intervals occupied by exactly calculable energy levels are depicted in figures 1 and 2.

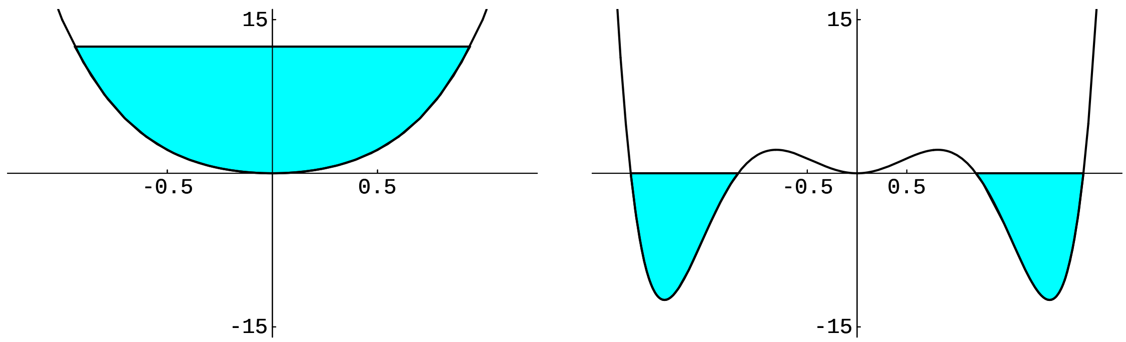

Figure 1: The form of the potential $V(x) = x^2(ax^2+b)^2 - acx^2$ (left picture) and its dual $\bar{V}(x) = x^2(ax^2-b)^2 - acx^2$ (right picture) for $a = 1$, $b = 3.3$ and $c = 3.5$. The gray domains correspond to the parts of spectra occupied by the exactly calculable levels. Here $[E_{min}, E_{max}] = [0, 12.37]$ and $[\bar{E}_{min}, \bar{E}_{max}] = [-12.37, 0]$.

5 The classical limit

It is naturally to ask what is the classical analogue of the phenomenon of quasi-exact solvability of models (2.1)? As far as we know, this question has never been raised in the literature. To answer it, we should understand what is the difference between the solutions of the classical equation of motion corresponding to the cases with $E \in [E_{min}, E_{max}]$ and $E \notin [E_{min}, E_{max}]$. For this it is very helpful to use the duality property.

First note that for small $\hbar$ the spacing between the neighbouring exactly calculable energy levels is of order of $\hbar$. If $\hbar \to 0$ then the lowest exactly calculable levels in models (3.2) and (3.3) approach the absolute minima of potentials $V(x)$ and $\bar{V}(x)$. For this reason, the classical limit of the relation (4.6) reads

$$[E_{min}, E_{max}] = [\min V_0(x), -\min \bar{V}_0(x)], \quad [\bar{E}_{min}, \bar{E}_{max}] = [\min \bar{V}_0(x), -\min V_0(x)], \quad (5.1)$$

where by $V_0(x)$ and $\bar{V}_0(x)$ we denoted the classical limits of the potentials $V(x)$ and $\bar{V}(x)$.

For determining the classical potentials remember that the original ones contain two arbitrary parameters $\hbar$ and M. Therefore, the result of classical limit will depend of how fast M tends to infinity if $\hbar$ tends to zero. This forces one to consider separaely two cases: 1) $\hbar \to 0$ and $c \to 0$ and 2) $\hbar \to 0$ and $c \neq 0$. These limits we shall call respectively the *soft* and *hard* classical limits.

The soft classical limit. In the soft classical limit the potentials (3.2) and (3.3) take the form

$$V_0(x) = x^2(ax^2+b)^2, \qquad \bar{V}_0(x) = x^2(ax^2-b)^2. \quad (5.2)$$

In this case we have

$$\min V_0(x) = 0, \qquad \min \bar{V}_0(x) = 0, \quad (5.3)$$

and therefore, the parts of the spectrum occupied by the exactly calculable levels are collapsing into a single (ground state) energy level $E = \bar{E} = 0$. For this reason, it is naturally to

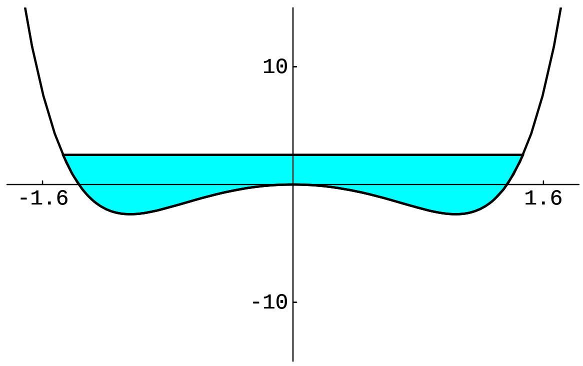

Figure 2: The form of the self-dual potential $V(x) = \bar{V}(x) = a^2x^6 - acx^2$ for $a = 1$ and $c = 3.5$. The gray domain corresponds to the part of the spectrum occupied by exactly calculable levels. Here $[E_{min}, E_{max}] = [-2.52, +2.52]$.

expect that the classical analogue of the phenomenon of quasi-exact solvability should lie in a comparative simplicity of the ground state solution ($E = \bar{E} = 0$) for the model (3.2), (3.3) with respect to all other solutions ($E > 0$).

In order to make sure that this is really so, consider, for example, the expression for the subbarier classical action $S(x)$ in the model (3.2). Remember, that function $S(x)$ is a natural classical counterpart of the quantum wavefunction $\Psi(x)$ and is related to it by the formula $\Psi(x) \approx e^{-S(x)/\hbar}$. Solving the corresponding Hamilton – Jacobi equation we obtain

$$S(x) = \int dx\sqrt{x^2(ax^2+b)^2 - E}, \qquad E \geq 0. \tag{5.4}$$

In some sense this is an explicit expression which, after evaluating the integral, can be reduced to the combination of tabulated elliptic integrals. Let us however consider the particular case with $E = 0$ when some analogue of quasi-exact solvability is expected. In this case the integral (5.4) can be evaluated explicitly and the result takes a very elegant and simple form

$$S(x) = \frac{ax^4}{4} + \frac{bx^2}{2}, \qquad E = 0. \tag{5.5}$$

We see that this case is distinguished from all other cases by the fact that the action function is in this case regular, without any branch - points and cuts! Comparing the special solution (5.5) with the general one (5.4), we can say with confidence that the former has all rights to be called "exact solution".

The hard classical limit. In the hard classical limit the potentials of the models (3.2) and (3.3) take the form

$$V_0(x) = x^2(ax^2+b)^2 - acx^2, \qquad \bar{V}_0(x) = x^2(ax^2-b)^2 - acx^2. \tag{5.6}$$

The number of exactly calculable energy levels tends to infinity and the spacing between them tends to zero. The sets of former quantum exactly calculable energy levels transform into the segments of a continuous spectrum given by the formula (5.1). The subbarier classical action now reads

$$S(x) = \int dx\sqrt{x^2(ax^2+b)^2 - acx^2 - E}. \tag{5.7}$$

It is not difficult to see that, in the complex x-plane, the function $S(x)$ has six square-root-type branch point singularities corresponding to the roots of the sixth-order algebraic equation $x^2(ax^2+b)^2 - cax^2 = E$. These singularities are connected pairwise by cuts. A simple analysis shows that if $E \in [E_{min}, E_{max}]$, then all the six singularities lie on the real and imaginary x-axes. However, if $E \notin [E_{min}, E_{max}]$, then there are singularities lying outside the axes. The location of these singularities and the corresponding cuts are depicted in the figure 3.

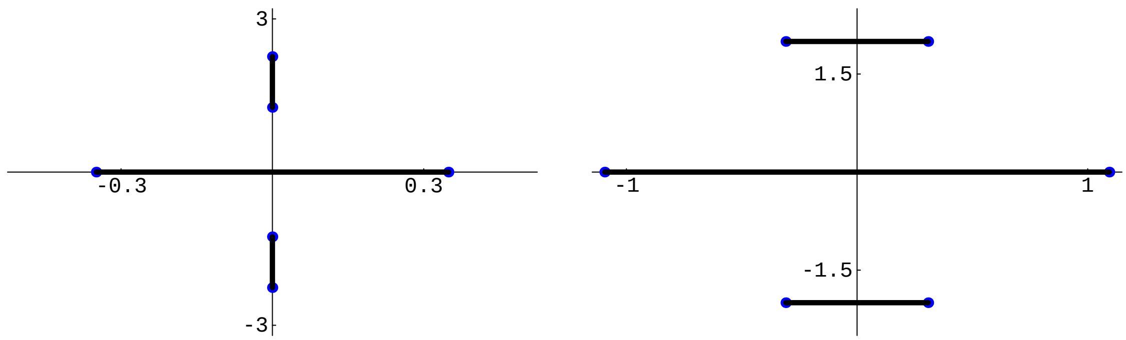

Figure 3: The complex x-plane for the classical action $S(x)$ (formula (5.7)). Here, as before, $a = 1$, $b = 3.3$, $c = 3.5$ and $[E_{min}, E_{max}] = [0, 12.37]$ is an interval occupied by the former exactly calculable energy levels. In the left picture $E = 1 \in [E_{min}, E_{max}]$, while in the right picture $E = 20 \notin [E_{min}, E_{max}]$. We see, that the disctribution of singularities and cuts for these two cases is different.

So, we see that there is some qualitative difference between the solutions of classical equations corresponding to the former exactly calculable and exactly non-calculable energy levels. At the same time, there is no difference between the complexity of these solutions. This is quite obvious, because if $M \to \infty$ the algebraic equations determining the exactly calculable states in quantum quasi-exactly solvable models become infinitely complicated and the difference between them and any other states dissapears.

6 The duality and orthogonal polynomials

The wavefunctions corresponding to exactly calculable energy levels in models (3.2) and (3.3) can be represented in the form

$$\begin{aligned} \Psi_m(x) &= x^p P_m\left(\frac{x^2}{\hbar}\right) \exp\left[-\frac{1}{\hbar}\left(\frac{ax^4}{4} + \frac{bx^2}{2}\right)\right], \\ \bar{\Psi}_m(x) &= x^p \bar{P}_m\left(\frac{x^2}{\hbar}\right) \exp\left[-\frac{1}{\hbar}\left(\frac{ax^4}{4} - \frac{bx^2}{2}\right)\right], \end{aligned} \tag{6.1}$$

where $p = 0, 1$ is the parity of the solution, and $P_m(t)$, $m = 0, 1, \ldots, [M/2]$ are certain polynomials of degree $[M/2]$. From the condition of the orthogonality of wavefunctions

$$\int_{-\infty}^{+\infty} \Psi_m(x)\Psi_n(x)dx = 0, \quad \int_{-\infty}^{+\infty} \bar{\Psi}_m(x)\bar{\Psi}_n(x)dx = 0, \quad m \neq n \tag{6.2}$$

it follows the orthogonality of polynomials $P_m(t)$ and $\bar{P}_m(t)$:

$$\int_0^\infty P_m(t)P_n(t)\omega(t)dt = 0, \quad \int_0^\infty \bar{P}_m(t)\bar{P}_n(t)\bar{\omega}(t)dt = 0, \quad m \neq n \tag{6.3}$$

with the weight functions

$$\omega(t) = t^{p-\frac{1}{2}}e^{-\frac{at^2}{2}-bt}, \quad \bar{\omega}(t) = t^{p-\frac{1}{2}}e^{-\frac{at^2}{2}+bt}. \tag{6.4}$$

Note now that the relation (4.1) for wavefunctions implies the analogous relation for the corresponding polynomials

$$P_m(t) \sim \bar{P}_{M-m}(-t) \tag{6.5}$$

whose comparison with (6.3) gives

$$\int_0^\infty P_m(t)P_n(t)\omega(t)dt = 0, \quad \int_{-\infty}^0 P_m(t)P_n(t)\omega(t)dt = 0, \quad m \neq n. \tag{6.6}$$

So, we see that polynomials $P_m(x),\ m = 0, 1, \ldots, [M/2]$ form an orthogonal system with the weight function $\omega(x)$ on two different intervals $[-\infty, 0]$ and $[0, \infty]$.

Note that the existence of two intervals on which the polynomials $P_n(x),\ n = 0, \ldots, [M/2]$ form an orthogonal system is closely related to the facts that all these polynomials are of the same degree $[M/2]$ and their number $[M/2]+1$ is finite. Indeed, in this case, the total number of coefficients of these polynomials (excluding the leading ones which can be considered as normalization factors) is $[M/2]([M/2] + 1)$. Requiring the orthogonality of polynomials on a certain interval, we obtain $[M/2]([M/2] + 1)/2$ equations for their coefficients. In this case the number of equations is only one half of the total number of unknowns which prevents one from determining these polynomials uniquely. However, if we require the orthogonality of these polynomials on two different intervals, then the number of equations will exactly coincide with the number of unknowns, and the polynomials can be determined uniquely. Note that this property is typical for all non-standard orthogonal polynomials associated with quasi-exactly solvable models from ref. [3].

7 Conclusion

The duality property of the simplest quasi-exactly solvable model (2.1) and its applications discussed in sections 3 – 6 can easily be extended to all one-dimensional quasi-exactly solvable models listed in ref. [3]. In order to demonstrate this, remember that these models are associated with the class of equations

$$\omega\left\{(t+\bar{a})t(t-a)\frac{\partial^2}{\partial t^2} + (At^2 + Bt + C)\frac{\partial}{\partial t} - M(M + A - 1)\right\}P(t) = EP(t) \tag{7.1}$$

(or their degenerate forms) having $M + 1$ polynomial solutions for any M. The transition of equation (7.1) to the Schrödinger form can be performed after introducing new variables x and new functions $\Psi(x)$ by the formulas

$$x = \int \frac{dx}{\sqrt{-\omega(t+\bar{a})t(t-a)}}. \tag{7.2}$$

and

$$\Psi(x) = P(t)\left(\frac{dx}{dt}\right)^{\frac{1}{2}} \exp\left\{\int \frac{At^2 + Bt + C}{\omega(t+\bar{a})t(t-a)}dt\right\} \tag{7.3}$$

in which the variable t belongs to the interval $[0, a]$ (for $\omega > 0$) or to the interval $[a, \infty]$ (for $\omega < 0$). The set of parameters $\omega, a, \bar{a}, A, B, C$ determines the form of the resulting potential. From (7.1) it immediately follows that the transformation $t \to -t$, $E \to -E$ reduces the quasi-exactly solvable model characterized by the set of parameters $\{\omega, a, \bar{a}, A, B, C\}$ to another model characterized by the set $\{\omega, \bar{a}, a, A, -B, C\}$. This transformation is nothing else than the generalization of the duality transformation for model (2.1).

It would be an interesting problem to investigate Lie algebraic origin of the duality property of quasi-exactly solvable models and find a multi-dimensional generalization of this phenomenon. We intend to consider these questions in fortcoming publications.

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